Heat Transfer to a Power-Law Fluid Flowing Between Parallel Plates

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The problem of heat transfer to flowing non-Newtonian fluids is of great importance in the processing of polymer solutions and melts. The classical papers of Bird (1955) and Toor (1957, 1958) dealt with heat transfer to non-Newtonian fluids in circular conduits. Subsequent investigations by Forsyth and Murphy (1969), Kim and Collins (1971), and Smorodinskii and Froishteter (1971) examined various aspects of this problem. Heat transfer to a parallel plate system has not had much attention despite its significance in polymer processing as, for example, in extrusion of polymer melts through a wide slit die or flow through a shallow channel melt screw pump. The only previous studies are those of Tien (1962) and Suckow et al. (1971). Tien used an approximate velocity profile to solve an extension of the Graetz-Nusselt problem to a power-law fluid flowing in a parallel plate system. This same problem was also solved by Suckow et al. with the difference that an exact velocity profile rather than an approximate one was used. However, in both papers the effect of viscous heating was not taken into account despite the fact that it was shown to be important in the heat transfer studies through circular conduits (Bird, 1955; Toor, 1957, 1958) and for Couette flow in parallel plate systems (Tien, 1961; Gavis and Laurence, 1968).

MATHEMATICAL PROCEDURE

For the purpose of mathematical analysis the following assumptions are made:

1. The fluid is incompressible and obeys the Ostwald-De Waele power-law with a constant consistency index m and constant thermal conductivity:

$$\tau = m \left| \frac{dU_x}{dy} \right|^{n-1} \frac{dU_x}{dy} \tag{1}$$

For pseudoplastics $n < 1$

2. The flow is steady.
3. End effects with respect to the velocity profile are neglected.
4. Gravity forces are neglected.
5. There is no "slip" at the wall.
6. Both plates are at a constant wall temperature T_w.

When these assumptions are made the equation of conservation of momentum reduces to the form (for flow in the x-direction and $2b$ the distance between the plates)

$$0 = -\frac{dp}{dx} + \frac{d}{dy}\left(m \left| \frac{dU_x}{dy} \right|^{n-1} \frac{dU_x}{dy} \right) \tag{2}$$

This equation can be easily solved and the result can be expressed in terms of the maximum velocity

$$U_x = U_{\max} \left[1 - \left(\frac{y}{b} \right)^{\frac{n+1}{n}} \right] \tag{3}$$

The equation of conservation of energy is

$$C_p\, \rho\, U_x \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} + m \left| \frac{\partial U_x}{\partial y} \right|^{n-1} \left(\frac{dU_x}{dy} \right)^2 \tag{4}$$

or, by substituting U_x from Equation (3), becomes

$$C_p\, \rho\, U_{\max} \left[1 - \left(\frac{y}{b} \right)^{\frac{n+1}{n}} \right] \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} + m \left| \frac{\partial U_x}{\partial y} \right|^{n-1} \left(\frac{dU_x}{dy} \right)^2 \tag{5}$$

We introduce the following dimensionless variables

$$\eta = (y/b) \tag{6}$$

$$\theta = \frac{T - T_w}{T_0 - T_w} \tag{7}$$

$$\psi = \frac{kx}{\rho C_p\, b^2 U_{\max}} \tag{8}$$

Note that ψ can be expressed in terms of the Peclet number as follows:

$$\psi = \frac{2}{3} \frac{1}{Pe} \frac{x}{b} \tag{9}$$

The equation of energy in terms of the dimensionless variables becomes

$$\left(1 - \eta^{\frac{n+1}{n}} \right) \frac{\partial \theta}{\partial \psi} = \frac{\partial^2 \theta}{\partial \eta^2} + \beta\, \eta^{\frac{n+1}{n}} \tag{10}$$

where

$$\beta = \left(\frac{n+1}{n} \right)^{n+1} \frac{m\, U_{\max}^{n+1}\, b^{1-n}}{k(T_0 - T_w)} \tag{11}$$

The coefficient β can be thought of as the product of the factor $[(n + 1)/n]^{n+1}$ and a non-Newtonian Brinkman number:

$$\beta = \left(\frac{n+1}{n} \right)^{n+1} Br_n \tag{12}$$

The boundary conditions for the Graetz-Nusselt problem under consideration are:

$$\psi = 0 \quad \theta = 1 \tag{13}$$

$$\eta = 1 \quad \theta = 0 \tag{14}$$

$$\eta = 0 \quad \frac{\partial \theta}{\partial \eta} = 0 \tag{15}$$

Equation (10) with the boundary conditions (13), (14), and (15) was solved numerically by the method of finite differences. A finite difference network of step size $\Delta\psi$ in the ψ-direction and $\Delta\eta$ in the η-direction was set up and the derivatives were replaced by the finite difference approximations

$$\frac{\partial \theta}{\partial \psi} = \frac{\theta(\psi + \Delta\psi, \eta) - \theta(\psi, \eta)}{\Delta\psi} \quad (16)$$

$$\frac{\partial^2 \theta}{\partial \eta^2} = \frac{\theta(\psi, \eta + \Delta\eta) - 2\theta(\psi, \eta) + \theta(\psi, \eta - \Delta\eta)}{(\Delta\eta)^2} \quad (17)$$

After substitution into Equation (10) an expression was obtained for $\theta(\psi + \Delta\psi, \eta)$ as a function of $\theta(\psi, \eta)$, $\theta(\psi, \eta + \Delta\eta)$, and $\theta(\psi, \eta - \Delta\eta)$. Since $\theta = 1$ at $\psi = 0$, the dimensionless temperature profile could be calculated for each $\Delta\psi$-step downstream. To ensure convergence of the finite difference scheme to the true solution several step sizes and step size ratios along and across the flow field were utilized. The results presented in subsequent figures and tables were independent of step size within at least 3 significant digits.

The dimensionless flow-average (bulk) temperature was calculated by using Simpson's rule and the definition

$$\theta_b = \frac{\int_{-b}^{b} \theta(\psi, \eta)\, U_x(\psi, \eta)\, d\eta}{\int_{-b}^{b} U_x(\psi, \eta)\, d\eta} \quad (18)$$

The local Nusselt number was calculated from the definition

$$Nu = \frac{2bh}{k} = \frac{2}{\theta_b}\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=1}$$

where the gradient of the dimensionless temperature at the wall was estimated from the finite difference approximation

$$\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=1} = \frac{1}{6\Delta\eta}\left(-11\theta(\psi, 1) + 18\theta(\psi, 1 - \Delta\eta) - 9\theta(\psi, 1 - 2\Delta\eta) + 2\theta(\psi, 1 - 3\Delta\eta)\right) \quad (19)$$

The computations were carried out in McMaster's CDC 6400 computer.

RESULTS AND DISCUSSION

To check the applicability of the numerical method, calculations were made for the case $n = 1$ (Newtonian), $\beta = 0$. The bulk temperature as a function of ψ is shown in Figure 1. Our results are indistinguishable from the calculated data of Prins et al. (1950) who solved analytically the Sturm-Liouville problem for the Newtonian case. The limiting Nusselt number (for large ψ) in Prins' work was $Nu = 3.770$ and in the present work $Nu = 3.767$. There are some differences from the work of Suckow et al., apparently due to the use of only the two first eigenvalues in their solution.

The effect of the flow index n on the bulk temperature θ_b and the local Nusselt number is shown in Table 1. The bulk temperature and the local Nusselt number decrease as the flow index increases. These conclusions are in agreement with Tien's (1962) work.

The effect of viscous dissipation on the bulk temperature and the local Nusselt number is shown in Figure 2. For $\beta = 0$ the Nusselt number reaches an asymptotic value (3.966) at about $\psi = 0.3$. For $\beta > 0$ ($T_0 - T_w > 0$) there is a minimum in the curve and the Nusselt number increases further downstream because of the viscous heating. For $\beta < 0$ ($T_0 - T_w < 0$) the Nusselt number decreases continuously in the region $0 < \psi < 1$. This is due to a continuously increasing contribution of viscous heating to the heated fluid. Consequently, the viscous dissipation

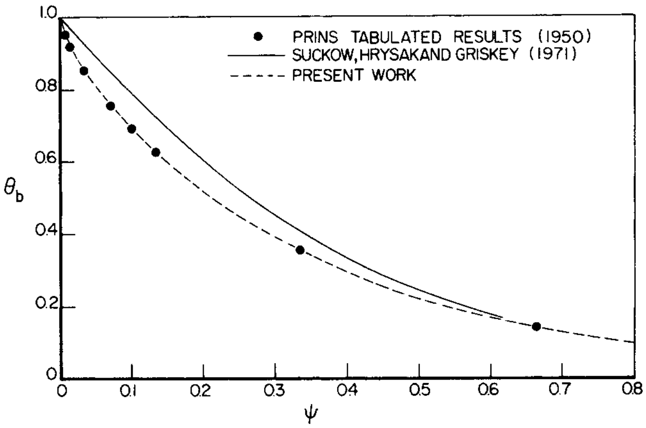

Fig. 1. Comparison of bulk temperature values obtained by various methods for a Newtonian fluid without viscous dissipation ($n = 1$, $\beta = 0$).

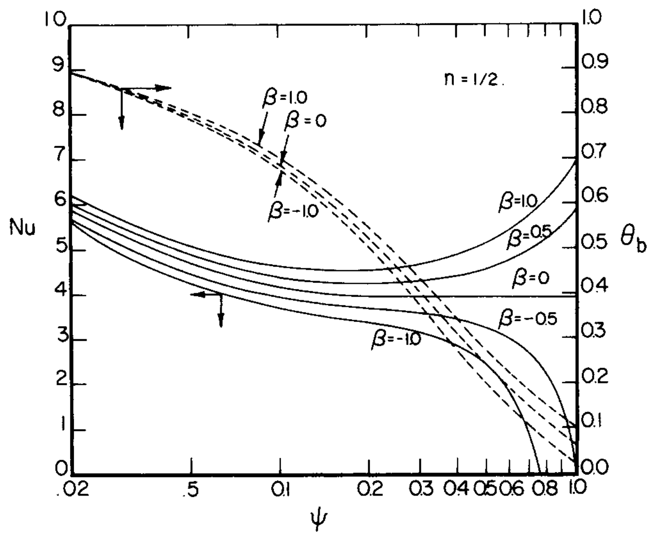

Fig. 2. Bulk temperature and local Nusselt Number as a function of dimensionless distance ψ.

TABLE 1. BULK TEMPERATURE AND LOCAL NUSSELT NUMBER FOR $\beta = 0$

	θ_b				Nu			
ψ	$n = 1/4$	$n = 1/2$	$n = 1$	$n = 2$	$n = 1/4$	$n = 1/2$	$n = 1$	$n = 2$
0.02	0.89	0.89	0.89	0.89	6.68	5.92	5.35	4.98
0.05	0.80	0.80	0.80	0.80	5.20	4.69	4.31	4.05
0.1	0.69	0.69	0.69	0.68	4.52	4.16	3.89	3.71
0.2	0.53	0.53	0.52	0.50	4.25	3.98	3.77	3.64
0.4	0.32	0.31	0.29	0.27	4.22	3.97	3.77	3.64
0.6	0.19	0.18	0.18	0.15	4.22	3.97	3.77	3.64
0.8	0.12	0.11	0.09	0.08	4.22	3.97	3.77	3.64
1.0	0.07	0.06	0.05	0.04	4.22	3.97	3.77	3.64

effects are important and should be taken into account in designing parallel plate systems in accordance with the results of this paper.

ACKNOWLEDGMENT

Financial assistance from the National Research Council of Canada is gratefully appreciated.

NOTATION

b = half the distance between the plates
Br_n = Brinkman number for a power-law fluid
C_p = specific heat
h = film heat transfer coefficient
k = thermal conductivity
m = consistency index
n = power-law index
Nu = Nusselt number
P = pressure
Pe = Peclet number
T = temperature
T_0 = temperature at $\psi = 0$
T_w = temperature at the wall
U_x = velocity in the x-direction
U_{max} = maximum velocity
x = coordinate in the direction of the flow
y = coordinate perpendicular to the direction of the flow
β = viscous dissipation parameter, defined by Equation (11)
$\Delta\eta$ = step size in the η-direction
$\Delta\psi$ = step size in the ψ-direction
η = dimensionless variable defined by Equation (6)
θ = dimensionless temperature defined by Equation (7)
θ_b = bulk temperature, defined by Equation (18)
ρ = density
τ = shear stress
ψ = dimensionless distance, defined by Equation (8)

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Manuscript received May 12, 1972; note accepted July 12, 1972.

Comparison of Various Ways of Model Building of a Regenerator

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For regenerators the dimensionless heat balances of the solid and gas, under usual assumptions, result in the following set of linear partial differential equations:

$$\frac{\partial S(z,t)}{\partial t} = G(z,t) - S(z,t) \quad (1)$$

$$\frac{\partial G(z,t)}{\partial z} = S(z,t) - G(z,t) \quad (2)$$

When the entrance gas temperature is constant and equal to the zero point of our temperature scale and when the initial solid temperature has the same value for all z, the normalized boundary conditions can be written as

$$S(z,t)|_{t=0} = 1 \quad (3)$$

$$G(z,t)|_{z=0} = 0 \quad (4)$$

The basic equations underlying many calculations in the